Misconceptions in Special Relativity – An Argument for Augmentation of the Modern Physics Course

By

Risley W. Mabile

Advisor: Dr. Ron Pepino

Science Education Advisor: Dr. Bromfield-Lee

A thesis submitted to the undergraduate faculty in fulfillment of the thesis

requirement for the completion of the honors program and degree of

Bachelor's of Science in Chemistry

Florida Southern College

Lakeland, Florida

2018

TABLE OF CONTENTS

Acknowledgments

First and foremost, I would like to thank Dr. Ron Pepino for his guidance and patience in this thesis project. Dr. Pepino is responsible for my decision to pursue this particular topic for my thesis, as well as being a friend and mentor who has constantly pushed me to work to improve myself over the course of this project. This honors thesis topic is separate from my chemistry research, and regards a topic that was far from my specialty. The ambition to proceed with this topic as opposed to staying within my major was largely fueled by the love of physics and intellectual curiosity that Dr. Pepino has cultivated in me. To him I am forever grateful.

Second, I would like to thank Dr. Bromfield-Lee for her advice and input on this project. Her expertise in constructing and distributing surveys proved invaluable over the course of the project.

Next, I would like to extend my gratitude and appreciation to all of the faculty of the Chemistry, Biochemistry, and Physics department for their support and wisdom throughout my undergraduate career. In particular, I would like to thank Dr. Le for his wisdom, guidance, and never-ending desire to help me improve as a scientist, society member, family member, and student. Without his caring advice and wise perspective, I could very easily be in a much different place at the end of my undergraduate career.

To always being continuous, but occasionally being non-differentiable.

Chapter 1

1.1 Introduction

On September 26, 1905, Albert Einstein published his work entitled "On the Electrodynamics of Moving Bodies", where he reconciled the inconsistencies between Maxwell's equations and Newtonian mechanics [1]. While this theory had implications that would take years to be accepted be the physics community, Einstein's work did not cease there. Shortly after publishing his special theory of relativity, Einstein began work to incorporate gravity into his theory – generalizing his theory of relativity. Einstein's work culminated 10 years later when he presented what are now known as Einstein's field equations at the Prussian Academy of Science in 1915. At the end of that same year, Einstein would publish his paper "The Foundation of the General Theory of Relativity" [2].

Einstein's field equations are a system of 16 partial, non-linear, coupled differential equations and as such are incredibly difficult to solve explicitly [3]. It is for this reason that General Relativity (GR) is never rigorously discussed in undergraduate and is not mandatory for graduate school programs. In fact, many physicists are never exposed to an honest discussion of GR. The mathematical machinery necessary for a rigorous discussion of GR to take place is no trivial obstacle to overcome. Instead,

undergraduate physics students are introduced to special relativity in an undergraduate Modern Physics course, which is a course designed to introduce sophomore physics majors to the most current understanding of the physical nature of the universe.

The Modern Physics course is typically taught in thirds: an introduction to special relativity, the history of the atom, and an introduction to quantum mechanics. In the first third of this course, the discussion is exclusively regarding special relativity. There are many interesting implications of special relativity that are discussed during this portion of the course including length contraction, mass-energy equivalence, relativity of simultaneity, the speed of causality, and time dilation. This conversation of relativistic mechanics usually culminates with what is known as the twin paradox [4,5].

The presentation of the twin paradox in Modern Physics ignores acceleration and assumes one of the twins can instantaneously accelerate to some large fraction of the speed of light. This ignores which twin is inertial and which undergoes acceleration and is ultimately what leads to the "paradox". In other words, the simplifying of this scenario for the discussion of time dilation is what leads to the so-called paradox of the problem. After the time dilation conversation has concluded, there is a very unsatisfactory clarification that the spaceship twin is not, in reality, inertial and as such the paradox is resolved. This presentation of the twin paradox is canonical; in

fact, no Modern Physics textbooks to date include a discussion of acceleration in the context of special relativity [4,5].

As hinted at above, GR cannot be rigorously discussed at this level due to the prohibitive difficulty of the mathematics involved; however, many Modern Physics textbooks will include a qualitative discussion of GR and of spacetime. It is at this point that many Modern Physics textbooks state the Einsteinian equivalence principle, *"The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime,"* which is often phrased by educators as the "local equivalence of gravity and acceleration" [6].

It has been noted by relativists, such as Sean Carroll, that there is a misconception amongst physicists that special relativity only applies to inertial situations, and any physical situation that involves acceleration must be reconciled through the use of GR [7]. In his textbook, *An Introduction to General Relativity: Spacetime and Geometry*, Sean Carroll states, "The notion of acceleration in special relativity has a bad reputation, for no good reason". The true statement is that the mass in question must be set up within an inertial coordinate system, but once the coordinate system is setup, any non-inertial trajectory of the mass is fully within the capabilities of special relativity. What makes special relativity *special* is that the

spacetime must have no curvature. In other words special relativity holds true in flat spacetime.

Unfortunately, a literature search does not reveal any hard evidence that this misconception actually exists, or amongst whom it exists. One Modern Physics textbook even incorrectly states, "Special relativity is concerned only with inertial frames of reference, that is, frames that are not accelerated" [8]. One would expect that if this misconception exists, it will be most prevalent in those physicists who have the smallest amount of experience with relativistic mechanics. For many fields, physicists are not exposed to relativity whatsoever with the exception of a Modern Physics course. If this is true, it would be expected that physicists in fields such as statistical mechanics or atomic, molecular, and optical (AMO) physics would have a higher prevalence of this misconception than those of relativity or particle physics where relativistic mechanics is employed more frequently.

1.2 Method

Constructing the survey

In an effort to quantitatively investigate if and where this misconception exists, a survey was constructed carefully to probe at three misconception-rich areas in statistical mechanics, classical mechanics, and quantum mechanics. This was done in order to mask the goal of the survey from the participant in an effort to reduce bias.

The full list of questions on the survey can be seen in Appendix A. The question that

we were particularly interested in on this survey is the following:

With regards to time dilation, what minimal level of theory is needed to solve problems
involving relativistic acceleration?
❖ Special Relativity.
❖ Special relativity is not capable of dealing with acceleration. General relativity is
   required.
❖ A combination of both special and general relativity is needed together.
❖ Cosmological relativity, because the boundary conditions of the universe are
   relevant.

The third option is a slightly absurd answer because special relativity is, in fact, a

special case of general relativity. The last option, regarding cosmological relativity, is

not adding anything new to the possible answers either, as "cosmological relativity"

refers to GR on cosmological scales. The nature of some of the misconception prone

areas in this survey is that there may be more than one correct answer depending on

the argument. However, that is not the case with this question. There is only one correct

answer – special relativity.

Survey distribution

To ensure ethical data collection, an application was submitted to the Florida

Southern College Human Subjects Institutional Review Board. The application was

reviewed by a committee member and approved.

To distribute the survey, emails were sent from the Principal Investigator's email address with the following text:

To whom it may concern,

I am a physics professor at Florida Southern College. As part of my undergraduate student's honors thesis, we have formulated a survey with the goal of identifying misconceptions in physics at all levels. This survey is meant to be administered to whoever is willing to participate at the undergraduate, graduate, and post-doctoral levels as well as faculty who are willing to participate at your institution. This is a very quick survey that should take no longer than five minutes. The survey consists of four questions that probe misconceptions in classical, quantum and statistical mechanics, as well as relativity theory. As outlined in the attached consent form, we would like each anonymous participant to disclose their level of education and possibly their field of research. This will help us determine which misconceptions exist at which levels of education and/or specialty.

The link below will take you directly to a consent form, which must be signed to gain access to the survey, as well as the survey itself.

Thank you for your time and consideration.
Sincerely, Asst. Prof. Ron Pepino

The consent form is attached to this email if you have interest in reviewing it before administering the survey within your department.

Prior to taking the survey, participants were required to read the informed consent form below from within a google form, and sign their understanding and agreement to the conditions of the survey prior to receiving a link to the survey.

Florida Southern College
Informed Consent Information

Project Title: Persistent Misconceptions in Physics

Principal Investigator: Dr. Ron Pepino

Phone: (863)680-3776   Email: rpepino@flsouthern.edu

Department: Chemistry, Biochemistry, Physics

Hello, I am researcher at Florida Southern College. You are being invited to participate in a research study regarding common misconceptions in physics. The goal of this study is to identify which misconceptions persist at different levels of education including: undergraduate, graduate and postdoctoral. In this particular study, we are concerned with topics in classical mechanics, relativistic dynamics, quantum mechanics and thermal physics. The anonymous data collected in this survey will be used to develop "guided learning" or "case-study" teaching approaches that can be administered in the appropriate courses. This data may also become part of physics education publication.

As part of this study, we ask you to complete a brief four question survey that should take roughly 5-10 minutes. The survey will be multiple choice, with a section to briefly provide a written response to justify your answer.

Since the goal of this survey is to obtain information about misconceptions, please refrain from using outside sources to answer these questions; simply answer each question based on your current understanding of each topic.

The data collected will be stored in a secure file, and your privacy and research records will be kept confidential to the fullest extent of the law. Any personal information that accompanies the data will be removed immediately upon reception. Your signing of this consent form is in no way linked to the data obtained in the survey. Furthermore, your institution affiliation is in no way traceable to your responses to the survey questions. Thus, there are no risks associated with taking this survey. Only authorized research personnel, employees of the Department of Health and Human Services, and the FSC Institutional Review Board may inspect the records from this project.

Your decision to participate is completely voluntary and there is no monetary compensation for taking this survey.

If you have any questions about this study, contact Dr. Ron Pepino at the phone number or email at the top of this form. If you have questions about your rights as an individual taking part in a research study, you may contact the Chair of the Florida

Southern College Institutional Review Board at (863-680-6205) or the FSC Vice President for Academic Affairs (863-680-4124).

Data collection and analysis:

The survey link took the participants to a google form with the demographic questions mentioned in the consent form, as well as the survey questions. The participants had the option to qualify their answers if they felt inclined, but this was left optional. As for the questions of the survey, responding to each question was required to be able to submit the survey. Data analysis was performed within Microsoft Excel.

1.3   Results

Table 1.1: Table showing the number of participants from each field of physics and their percentage of the total sample population.

Specialty	# of participants	Percentage of pop.
Astrophysics	5	8.3%
AMO	5	6.9%
Biophysics	3	4.1%
Condensed Matter	14	19.4%
GR or Cosmology	9	12.5%
Particle/nuclear	22	30.5%
PER	4	5.5%
Other	9	12.5%

Table 1.2: Table showing the demographics of each subpopulation of participants.

Field	# of Faculty	# of PhD/Postdocs	# of Master's program students	# of undergraduates
Astrophysics	0	4	2	0
AMO	2	0	0	3
Biophysics	2	0	0	1
Condensed Matter	8	4	0	2
GR/Cosmology	5	3	1	0
Particle/Nuclear	13	5	0	4
PER	4	0	0	0
Other	4	3	0	2

Table 1.3: Table showing the percentage of each subpopulation which answered the relativity question correctly.

Specialty	Percent correct
Astrophysics	80%
AMO	40%
Biophysics	0%
Condensed Matter	57.1%
GR/Cosmology	66.7%
Particle/Nuclear	45.5%
PER	50%
Other	55.6%
Total Population	52.1%

Table 1.4: Table showing the percentage of correct answers by professional level.

Education level	Percent correct
Faculty	56.41%
PhD/Postdoc	36.84%
Masters	66.67%
Undergraduate	50%

Table 1.5: Comments made by participants as justification for their answer to the relativity question.

| Professional Level	Comment
Faculty	Special relativity applies only to constant velocity motion, in the absence of mass. Acceleration necessarily involves general relativity, but Mach's principle.
Faculty	Special relativity has no problem with acceleration; one just needs to stick to inertial frames.
Undergraduate	Special Relativity explains constant motion at speeds close to c, while general relativity explains non-uniform motion near c.
Master's program	The answer, "Both special and general relativity" doesn't make sense, as special relativity is a subset of general relativity. Special relativity alone cannot explain what happens during acceleration, only what comes before and after.
PhD program	Special relativity is a subset of general relativity, where spacetime is flat, i.e., where there is no acceleration
PhD program	Special relativity only deals with inertial reference frames, which are non-accelerating
PhD program	Special relativity is just for fast things
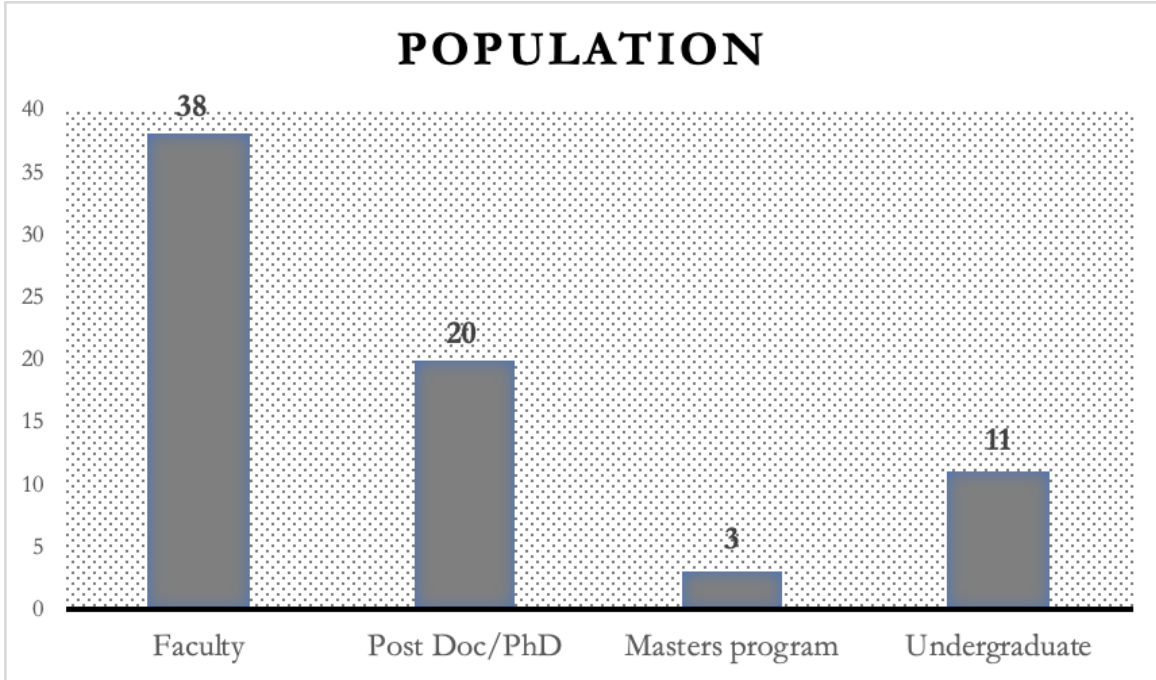
Figure 1.1: Population demographics of the participants of the survey. The total sample size was n=72. Institution faculty make up 38 out of the 72, post doc and PhD program students make up 20, master's program students were the least represented with 3 responses, and undergraduates make up the last 11 participants of the total.
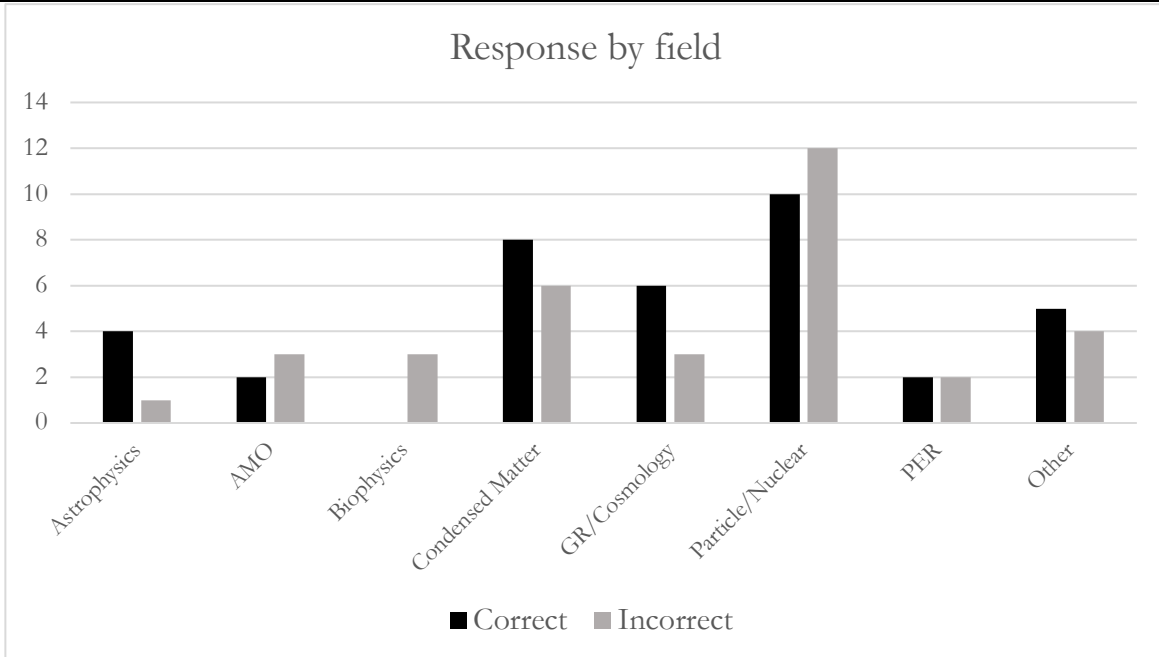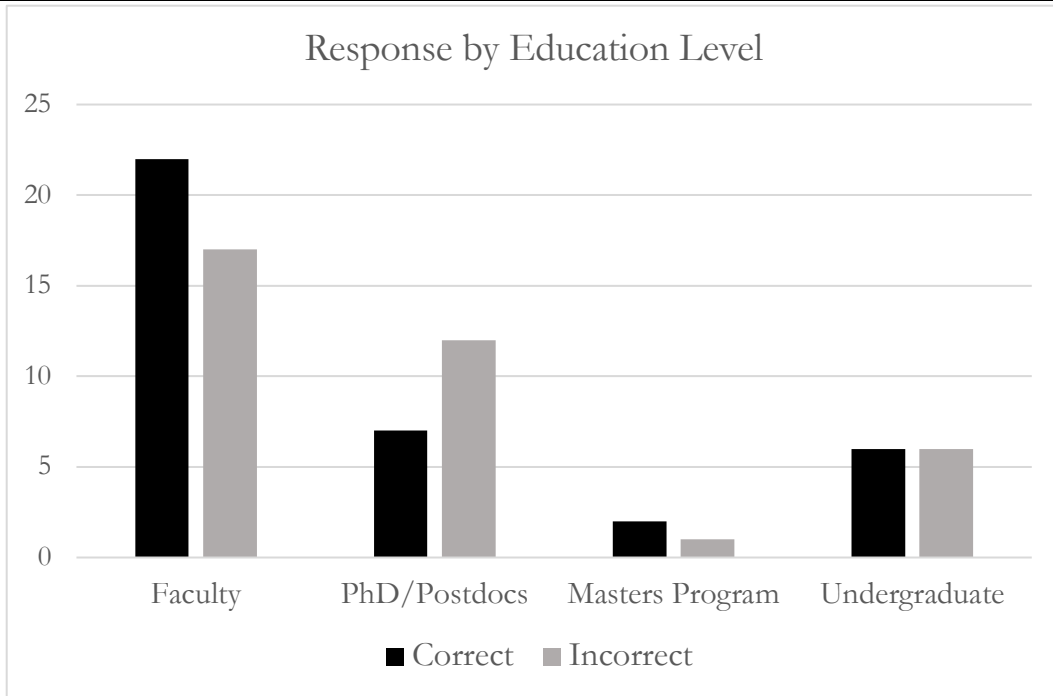

Figure 1.2: Column diagram showing the number and correct versus incorrect answers by field. There was a response from a faculty member in astrophysics whose response did not register within the google survey, and as such, there are 71 usable samples for this question.

Figure 1.3: Column diagram showing the number of correct and incorrect answers as a function of presentation level. The undergraduates were correct 50% of the time, while the PhD/Postdocs performed the worst out of any education level. The faculty were correct

1.4 Discussion

While this survey was sent to 62 institutions across the United States, a total of 72 responses were received (including responses from Harvard, MIT, and Caltech). The demographics (Figure 1.1) reveal that the largest professional level represented was faculty members at their respective institutions. The fields' of physics and their respective representation can be seen in Table 1.1. Table 1.2 shows the distribution of professional levels represented within each field of physics. It is worth mentioning that there was an error in the google form which caused one of the astrophysicists' responses

to be lost. The demographics were performed for n=72, but all subsequent analysis was performed with n=71. The largest field represented was nuclear/particle physics which consisted of 30.5% of the total population. The smallest field represented by this data is biophysics at 4.1% of the total population.

While this is not the ideal distribution of responses by field that we would have hoped to obtain, this is the nature of conducting surveys. While each field may not have as many participants as would be ideal, there is good diversity of representation in our sample and as such the information is still useful. Because of the nature of the information obtained from the participants, there are a few interesting ways to look at the data.

The first interesting way to analyze the data is by looking at the correct versus incorrect responses of the participants as a function of their field (Figure 1.2, Table 1.3). We would expect to see that the astrophysicists and particle physicists score the highest considering that both of these fields employ relativistic mechanics regularly. As shown in Table 1.3 and Figure 1.2, the astrophysicists were 80% correct; that is to say, only one of the astrophysicists answered incorrectly. It is worth noting that all of the astrophysicists who replied were graduate students (Table 1.2).

To the contrary of what was expected, only 45.45% of the particle/nuclear physicists answered correctly. While the subpopulation from particle/nuclear physics

includes 4 undergraduates, this is still surprising when one considers the level of use that relativistic mechanics receives within the field of particle physics. Furthermore, 60% of the subpopulation of particle/nuclear physicists are faculty members. From the data obtained in this survey, the particle/nuclear physics subpopulation is the most statistically representative of any of the subpopulations. It should be obvious that there is a significant problem when less than 50% of particle/nuclear physics faculty are able to answer such a simple question correctly. Arguably, their field should be the most familiar with the application of special relativity to non-inertial problems.

Looking at the entire population (Table 1.4), faculty members only answered this question correctly 56.41% of the time. The population average for all education levels was 52.1% correct. Based on this data, roughly half of the physics community is unaware of the ability of special relativity to deal with non-inertial situations.

Another portion of this data that is worth discussing are the comments that individuals left to justify their answers because these comments are telling of what participants are thinking when they answer correctly or incorrectly. Some of the comments made by individuals are seen in Table 1.5. From the comments, it is clear that the people who answered correctly are aware that special relativity is the subset of general relativity where there is no spacetime curvature – participants who commented along these lines all answered correctly that special relativity is capable on

its own. On the other hand, when this question was incorrectly answered, the justification was that special relativity is only for constant velocity motion at velocities near the speed of light. It can also be seen in Table 1.4 that when the participant knew what they were talking about, the non-sensical nature of two of the four answers was obvious to them (Table 1.4, line 5). The professional levels seen in Table 1.4 reveal that this misconception is persistent to the level of PhD candidates and faculty members.

It would seem obvious that this misconception is likely formed in a Modern Physics course where special relativity is only discussed in the inertial case, and acceleration is only discussed in the context of GR. This must be the case because the majority of physicists may never see relativity again in their career unless they go into a field where it is relevant. This would imply that there is a problem with the way that relativity is being discussed in Modern Physics.

1.5 Conclusions and future prospectus

While the data obtained in this study was not as ideal as one would prefer, the data seems to imply that the physics community is not as well aware of the capabilities of special relativity as they should be. This data also hints that the issue is a systemic one, as faculty may not truly understand what they are teaching at the level necessary to teach the subject. The rectifying of this misconception is not difficult, especially with the mathematical background every physicist must obtain over the course of their

undergraduate career. The only mathematical tools necessary to eradicate this misconception from the students' (or faculty members') mind is a background through Calculus 2, which is a course taken by most physicists their freshman year in college. Since Modern Physics is sophomore level course, there appears to be no good reason not to address this misconception head-on by solving non-inertial problems in a Modern Physics course. It would be prudent moving forward to identify and employ an expedient teaching method that can effectively rectify this misconception directly, and this will be a topic of future work.

1.6   References

[1]   A. Einstein, *Annalen Phys.*, 17, 891-921 (1905).

[2]   A. Einstein, *Sitzungsber.* 44, 778-799 (1915).

[3]   H. Heintzmann, *Zeitschrift Für Phys.* 228, 489-493 (1969).

[4]   J. Morrison, *Modern Physics for Scientists and Engineers,* Chap. 1, 1ˢᵗ ed., (Academic press, 2010).

[5]   S. T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers*, Chap. 1, 4th ed. (Brooks/Cole, 2013).

[6]   A. L. Harvey, *Ann. Phys.*, 29, 383-390, (1964).

[7]   S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, 1st ed., Chap. 1, Sec. 2 (Pearson, 2003).

[8]   R. Harris, *Modern Physics*, Second Edition, Chap. 1, (Pearson: Addison-Wesley, 2008).

1.7 Appendix A

All questions used in the survey are displayed below:

1. Which of the following explanations best describes the reason why the earth gets two ocean tides per day instead of just one?
   a. The non-uniform gravitational field of the moon pulls stronger on the side closest to the earth.
   b. A combination of the gravitational attraction of the oceans by both the sun and the moon.
   c. The non-uniform gravitational field of the moon coupled with the rotation of the earth about the center of mass of the earth-moon system.
   d. A non-uniform gravitational field of the moon coupled with the centripetal force due to the earth's rotation.
2. The Heisenberg Uncertainty Principle states that the position and momentum of a particle cannot both be known with 100% certainty at the same time. Which of the following best describes the reason behind this uncertainty?
   a. The act of measuring a quantum system affects the system itself.
   b. The wave-like nature of quantum mechanics.
   c. The principle is a fundamental postulate of quantum mechanics.
   d. Position and momentum are incompatible observable quantities.
3. With regards to time dilation, what minimal level of theory is needed to solve problems involving relativistic acceleration?
   a. Special Relativity.
   b. Special relativity is not capable of dealing with acceleration. General relativity is required.
   c. A combination of both special and general relativity is needed together.
   d. Cosmological relativity, because the boundary conditions of the universe are relevant.
4. Which of the following best describes entropy from a statistical mechanics perspective?
   a. It is a quantity related to the disorder in a system.
   b. It is related to the number of possible states of a system.
   c. It is the usable energy of a system.
   d. It is a consequence of a system that is not in thermal equilibrium.

Chapter 2

2.1 Introduction

The teaching strategies that are employed in education are constantly evolving as our understanding of how the brain processes information changes. Educational psychology has been at the cutting edge of the development of recent strategies for improved learning. For example, engaged learning – which is now a fairly widespread practice – is a product of educational psychology research and development [1]. A more recent development in educational psychology, dealing with misconceptions directly, is one that is making apparent the value of the quality of education vs. the quantity of material covered [2]. In physics education, dealing directly with these misconceptions is only recently becoming commonplace, even though the existence of many of these misconceptions has long been identified [2-6]. This is somewhat odd considering the frequency of counter-intuitive and misconception-rich topics covered in a physics course. As the teaching methods used in physics education continue to improve, it is vital that these misconceptions begin to be dealt with forthrightly – especially in first and second year courses when the foundation of physical understanding is being laid out.

Preconceived notions have been shown to plague some crucial topics in physics such as energy, gravity, and heat [7]. These are the misconceptions that a person often derives from their perception of the world around them without due scientific process or application of critical thinking. An example of this type of misconception is believing that underground water *must* flow in streams because that is how it flows on the surface [6]. Dealing with these misconceptions in the classroom is an excellent opportunity to guide a student in their development of critical thinking in the context of physics. The conceptual misunderstandings that are formed when a student first hears an explanation of material are often difficult to deal with. However, in dealing with these misunderstandings there is another opportunity for the teacher to guide the students' contextual critical thinking development. One way that has shown strong promise in dealing with both of these types of misconceptions is through the use of case studies [8-9].

When a student is simply lectured *at* in a classroom instead of being engaged, they are more likely to either form misconceptions or not gain anything other than a grazing familiarity with the topic being covered. The case-study approach is another development out of the field of educational psychology that has recently gained a strong following, and with good reason. In introductory chemistry courses, for example, the use of case studies in the classroom has been shown to significantly improve both the

students' meaningful understanding of the material as well as their overall attitude towards chemistry. This application of the material to numerous cases and examples allows the generalization and abstraction of the material in the mind of the student [8]. This is an approach that has been used extensively in the field of physics education. It is crucial to mention that these case studies are done in the classroom with teacher-guided discussion as opposed to being assigned as homework problems. This helps ensure the students are working through the problems via discussion and critical thinking, as opposed to using a quick internet search.

Part of the struggle in developing course syllabi is determining what topics should be covered, and at what depth that material should be covered. In the context of the use of case studies, they are time-consuming to use in the classroom and take away from potential lecture time. However, a study performed by Sadler *et al.* showcased the benefits of a quality covering of introductory physics at the high school level, as opposed to a quantity approach (covering many topics more superficially) [10]. High school students who had courses that covered fewer topics, at greater depth, performed significantly better than their peers in first year undergraduate physics regardless of their mathematical background or the number of semesters of physics taken in high school. This benefit likely stems from the more adequate development of the students' contextual critical thinking skills [11]. While part of the benefit of covering

more topics in a course is the exposure that it gives the students, the evidence suggests that these courses should err more on the side of covering less topics - in more depth.

One potential place in physics that this approach could be further developed is in the discussion of special relativity in Modern Physics courses. As we showed in chapter 1, there exists a misconception around acceleration in special relativity that is both prevalent and persistent to the level of faculty members across numerous institutions including Harvard, MIT, and Caltech. This should be concerning, especially when one considers that nearly half of the faculty do not fully understand this topic.

Indeed, coordinates in special relativity must be set up such that they are inertial, but this does not rule out dealing with accelerated trajectories so long as the spacetime remains flat. In this chapter we will pick up at the ending point of many Modern Physics courses' discussion of special relativity and demonstrate the simplicity with which acceleration in special relativity can be discussed. Below we provide an example of a case study approach and how this can be used to rectify this misconception.

2.2   Background/Theory

To initiate a conversation about acceleration in special relativity, it is prudent to first motivate the spacetime interval rather than quote it. We will begin with a brief discussion of Euclidean geometry and Cartesian coordinate systems. Let us first define two coordinate systems in $\mathbb{R}^3$ with arbitrary relative orientation and origin. We will

define the first coordinate system $A \equiv \{(x, y, z) \in \mathbb{R}^2\}$ and the second coordinate system $B \equiv \{(x', y', z') \in \mathbb{R}^2\}$. Now consider any two arbitrary points in that Euclidean space. The Pythagorean theorem simply implies that the distance between any two arbitrary points according to coordinate system A is given by

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{1}$$

Similarly the distance in B is given by

$$d'^2 = \Delta x'^2 + \Delta' y^2 + \Delta' z^2 \tag{2}$$

Which straightforwardly implies

$$d^2 = d'^2 \tag{3}$$

Mathematically, we would say that this quantity is invariant under the transformation T: A→B. That is to say, no matter what orientation or origin is used for two coordinate systems, the distance between any two points, p_1 and p_2, remains invariant. This is generally true of any transformation between Cartesian coordinate systems in Euclidean space with dimension $\mathbb{R}^n$. With this definition, Euclidean space becomes metric space, and Equation 3 is known as the Euclidean metric.

As is covered early in a Modern Physics course, Cartesian distances are no longer invariant in relativity. This is exemplified by the notion of Lorentz contraction, where a stationary observer and an observer in a moving reference frame will disagree on the distance between two points. Similarly, observers in different reference frames will

disagree on the *length* of time as a result of time dilation. To merge space and time, we need to know what our analogous invariant quantity is.

We will now transition into our discussion of special relativity. We will pick up this discussion with a topic that is often well motivated and discussed in depth in a Modern Physics course, the Lorentz transformations. The Lorentz transformations are linear transformations that describe the transformation from one coordinate frame to another in Minkowski space. We will again define two coordinate systems where $K \equiv (x, y, z)$ are the cartesian spatial coordinates of the stationary frame and $K' \equiv (x', y', z')$ are the cartesian spatial coordinates of the moving frame. Then the Lorentz transformations are defined as follows:

$$
\begin{aligned}
x' &= \frac{x - vt}{\sqrt{1 - \beta^2}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - \left(\frac{vx}{c^2}\right)}{\sqrt{1 - \beta^2}}
\end{aligned}
\tag{4}
$$

Here v is the velocity of the moving frame K' relative to K, c is the speed of light, and $\beta = \frac{v}{c}$ is the relativistic factor. These transformations are employed heavily in a Modern Physics course, and as such it is a good starting point for the rest of this discussion.

We seek to find the invariant quantity similar to that in Equation 3 such that we can identify what quantity a stationary and moving observer would agree upon in

Minkowski space. If we incorporate the Lorentz factor $\beta = \frac{v}{c}$ into our transformations and consider the infinitesimal change they become

$$
\begin{aligned}
dx' &= \gamma(dx - vdt) \\
dy' &= dy \\
dz' &= dz \\
dt' &= \gamma\left(dt - \frac{vdx}{c^2}\right)
\end{aligned}
\tag{5}
$$

Let us guess that the invariant quantity we seek is of the form

$$
ds^2 = c^2 dt^2 - d\vec{r}^2
\tag{6}
$$

Multiplying the temporal transformation by a factor of c, squaring each transformation, and subtracting the position transformations from the temporal transformation we find

$$
c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2
\tag{7}
$$

$$
= \gamma^2\left(c^2 dt^2 + \frac{v^2 dx^2}{c^2} - 2vdxdt - c^2 dx^2 - v^2 dt^2 + 2dxvdt\right)
$$

$$
- dy^2 - dz^2
$$

Further simplifying we get

$$
c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2
\tag{8}
$$

$$
= \gamma^2\left(c^2 dt^2 + \frac{v^2 dx^2}{c^2} - dx^2 - \beta^2 c^2 dt^2\right) - dy^2 - dz^2
$$

Continuing with the simplification

$$
c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = \gamma^2[c^2 dt^2(1 - \beta^2) - dx^2(1 - \beta^2)] - dy^2 - dz^2
\tag{9}
$$

Finally we arrive at
$$
c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2
\tag{10}
$$

Therefore the invariant quantity in Minkowski space is of the form of Equation 6.

This relationship is a crucial result. This is what is known as the metric of a Lorentzian manifold – more commonly referred to as the spacetime interval. These are typically abbreviated by the following

$$\mathrm{ds'}^2 = \mathrm{ds}^2. \tag{11}$$

An important distinction should be made between the spacetime interval and four-dimensional Euclidean space. The sign difference between the time dimension and the spatial dimensions is not present in four-dimensional Euclidean space. One notable implication is that the spacetime interval between two separate events can be zero. This occurs when $\mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2 = \mathrm{c}^2\mathrm{dt}^2$, in other words when the velocity of an object is equal to the speed of light.

If a reference frame is defined as spatially stationary, that is dx=dy=dz=0, then the relationship between the two frames can be written as

$$\mathrm{c}^2\mathrm{d\tau}^2 = \mathrm{c}^2\mathrm{dt'}^2 - \mathrm{dx'}^2 - \mathrm{dy'}^2 - \mathrm{dz'}^2 \tag{12}$$

Where τ, which is referred to as the *proper time,* is the time seen by a clock in the stationary reference frame, K. This is useful because of the relative nature of the reference frames. This expression yields the relationship between the length of the worldlines of K and K'.

In Cartesian coordinates we are used to seeing the displacement vector as a 3-vector of the form

$$\mathrm{d}\vec{r} = (\mathrm{dx}, \mathrm{dy}, \mathrm{dz}) \tag{13}$$

However the spacetime interval arrives from a set of coordinates in Minkowski space where

$$\mathrm{d}\vec{R} = (\mathrm{cdt}, \mathrm{dx}, \mathrm{dy}, \mathrm{dz}). \tag{14}$$

Here $\mathrm{d}\vec{R}$ is known as the 4-vector. To maintain a distinction between the spatial coordinates and the temporal one, this is commonly referred to as 3+1 dimensional spacetime. This is useful because the spatial dimensions are bidirectional, but the temporal dimension is unidirectional. The path of displacement in Minkowski space is referred to as a world line. For the canonical presentation of the twin paradox, the worldlines are seen below in Figure 2.1.
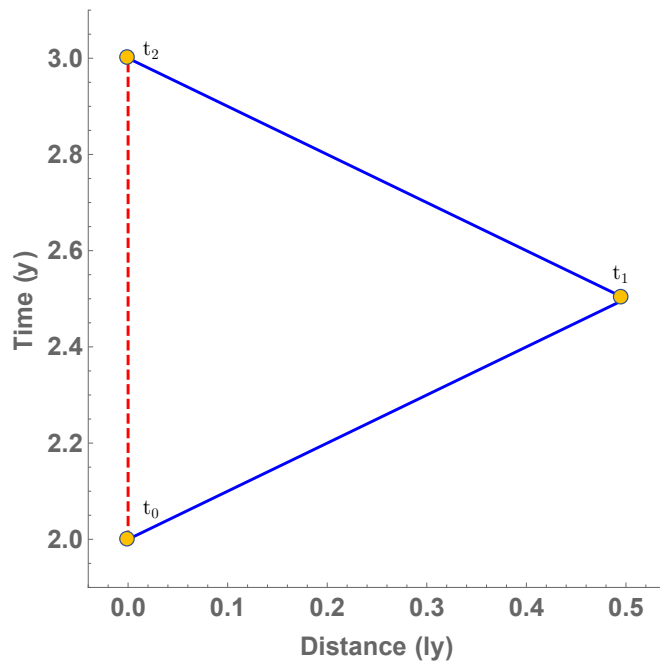
Figure 2.1: Spacetime diagram illustrating the world line of the earth twin (red, dashed) versus the spaceship twin (blue, solid) as seen in reference frame K.

2.3   Proposed conversation

The problem at the heart of the twin paradox lies at the points of acceleration. It should be intuitively obvious that undergoing infinite acceleration would turn a human and their space ship into a pancake. In the classical presentation of the twin paradox, this "pancaking" happens three times: Δv: 0c$\rightarrow$ 0.8c, Δv: 0.8c$\rightarrow$ $-0.8c$, and Δv: -0.8c$\rightarrow$ 0c (Figure 2.1).

Moving forward, it is necessary to show that undergoing a reasonable acceleration, say 9.8 m/s, would still allow for significant time dilation to be experienced by the twins. For convenience sake we will, of course, ignore the energy required to maintain that acceleration. Let us first simplify our physical scenario by asserting that dy=dz=0, yielding

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{\left(dx/dt\right)^2}{c^2}} \tag{15}$$

Where τ is the proper time, t is the time in the moving frame, and x is the spatial coordinate. This is the spacetime interval of the moving twin as seen from the reference frame of K. Now we will integrate over all infinitesimal changes, which allows us to stay within an inertial frame for each small change. Doing so yields the following:

$$\Delta\tau = \int_{t_i}^{t_f} \sqrt{1 - \frac{v^2}{c^2}}\, dt \tag{16}$$

We can now apply this expression to the canonical presentation of the twin paradox, without acceleration. Let the spaceship twin instantaneously accelerate to some fraction of the speed of light, v_1, in the x-direction at time t_0. At some time t_1, the twin instantaneously reverses their velocity. Finally, at time t_2, the twin instantaneously comes to rest back on earth. Mathematically:

$$\Delta\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v_1^2}{c^2}} \, dt + \int_{t_1}^{t_2} \sqrt{1 - \frac{v_2^2}{c^2}} \, dt \tag{17}$$

Here, the expressions inside each integral are constants, and the velocities are equal and opposite. Simultaneously recognizing that the constants are in fact the inverse Lorentz factor, the integral reduces to

$$\Delta\tau = \sqrt{1 - \frac{v_1^2}{c^2}} \int_{t_0}^{t_2} dt \tag{18}$$

$$\implies \Delta\tau = \gamma^{-1}\Delta t$$

$$\implies \quad \gamma = \frac{\Delta t}{\Delta\tau}$$

As a result of this computation, we have arrived at the formal relationship of the Lorentz Factor, γ, to the proper time. This showcases the utility of this presentation of the paradox – reducing it to an algebra problem.

- **The accelerated twin paradox**

Let us now define a new twin paradox situation. We will define the observer on the earth as being in the reference frame K, and the spaceship twin to be in the reference

28

frame K'. In this situation, the spaceship is passing earth at some time, t_0, with a velocity $v_0=0.8c$ relative to the stationary reference frame. At t_0 the twin passes the earth, waves at his earth twin, and synchronizes his watch with his twin. He then begins accelerating in the negative x-direction at $g=9.8$ m/s^2 (a non-pancaking acceleration). At some time, t_1, the twin passes the earth once again at $v_2=-v_1$ and continues in the negative x-direction. The spacetime diagram for this problem is given below in figure 2.2.
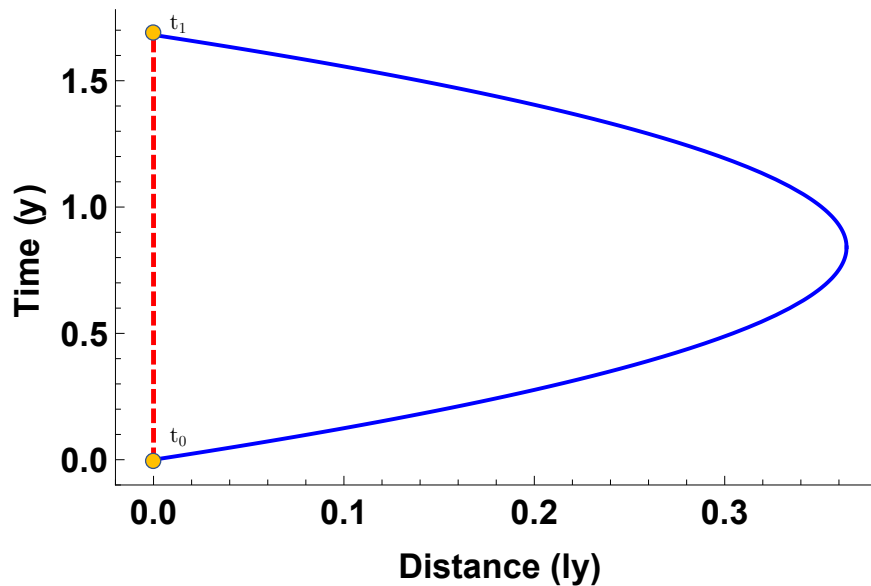

Figure 2.2: Spacetime diagram illustrating the worldline of the earth twin (red, dashed) versus the spaceship twin (blue, dashed) as seen in reference frame K.

To adjust our expression in Equation 15 for this scenario, the velocity simply becomes a kinematics problem. This should lead us to the following:

$$\Delta\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{(v_i - gt)^2}{c^2}}\, dt \tag{19}$$

Thus, this is now a Calculus 2 problem as opposed to an elementary algebra problem. Integrating this equation, we find

$$\tau(t) = \frac{c}{2g}\left[\left(\frac{v_i - gt}{c}\right) \sqrt{1 - \left(\frac{v_i - gt}{c}\right)^2} + \arcsin\left(\frac{v_i - gt}{c}\right) \right] \tag{20}$$

This is the solution to an accelerated twin paradox example as outlined above. However, this is also more generally the function that describes the relationship between the proper time of a stationary observer and the proper time experienced by a non-inertial traveler that is undergoing a constant acceleration in flat spacetime. To cater this solution to another example, the acceleration and initial velocity must of course be exchanged.

For a similar example, let the spaceship twin start on earth. At t_0, the twins synchronize their watches and the spaceship twin begins accelerating away from the earth at g until some time, t_1, at which point the direction of the acceleration is reversed. The twin continues undergoing this acceleration until some time t_2, where the acceleration reverts back to the original acceleration direction, landing the twin stationary softly back on earth at time t_3. The space time diagram for this scenario can be seen below in figure 2.3.
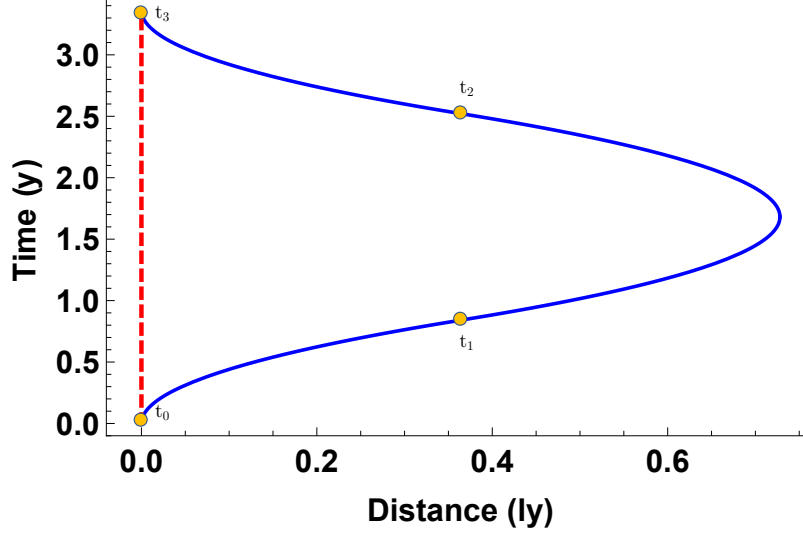
Figure 2.3: Spacetime diagram illustrating the worldlines of the earth twin (red, dashed) versus the spaceship twin (blue, solid) as seen from reference frame K.

The expression for this scenario is the following

$$\Delta\tau = \int_{t_0}^{t_1} \sqrt{1 - \left(\frac{gt}{c}\right)^2} \, dt + \int_{t_1}^{t_2} \sqrt{1 - \left(\frac{v_1 - gt}{c}\right)^2} \, dt \tag{21}$$

$$+ \int_{t_2}^{t_3} \sqrt{1 - \left(\frac{v_2 + gt}{c}\right)^2} \, dt$$

Where v_1 is the velocity at t_1, and v_2 is the velocity at t_2. This yields the solution

$$\tau(t) = \frac{c}{2g} \left[\arcsin\left(\frac{gt}{c}\right) + \left(\frac{gt}{c}\right) \sqrt{1 - \left(\frac{gt}{c}\right)^2} + \arcsin\left(\frac{v_1 - gt}{c}\right) \right.$$

$$+ \left(\frac{v_1 - gt}{c}\right) \sqrt{1 - \left(\frac{v_1 - gt}{c}\right)^2} + \arcsin\left(\frac{v_2 + gt}{c}\right) \tag{22}$$

$$\left. + \left(\frac{v_2 + gt}{c}\right) \sqrt{1 - \left(\frac{v_2 + gt}{c}\right)^2} \right]$$

31

2.3 Discussion

As previously mentioned, the conversation proposed above is not discussed in any Modern Physics textbooks. While the mathematics involved for this discussion are more advanced than the ones required for the canonical presentation of the twin paradox, they are also mathematical tools that will be discussed in every physics majors' undergraduate career. Often, the mathematics involved are taught in the physics majors sophomore year. For example, the integration technique used above is trigonometric substitution; this is often taught in second-semester calculus and is a very useful technique to be familiar with because of its relevance in upper-level physics courses. As such, an argument that the mathematics for this discussion are too involved is a poor one.

Furthermore, this discussion can take the place in a single lecture, or portion of a lecture. The implied argument here is that having this discussion in a Modern Physics course addresses the misconception head on by demonstrating that special relativity is fully capable of dealing with non-inertial scenarios. Of course, the emphasis should be placed on the fact that special relativity describes all dynamics that take place on a Lorentzian manifold. This should decrease the prevalence of this misconception. An example of a case study that could be used to deal with this misconception directly can be seen in appendix B.

The case study approach is useful because the students are guided through the worksheet by the professor, and as such must contend with each step towards the conclusion. Unfortunately, being a student at a small institution makes generating statistics to verify the use of case studies untenable; however, the evidence for the success of such methods when applied correctly is indisputable [8,9].

In a more broad sense, this work combined with the physics education literature suggests a subtle critique of the design of a typical Modern Physics course: the evidence supports the notion that covering fewer topics in greater detail leads to better outcomes for physics students [10]. Certainly, this work suggests that simply spending one additional lecture on relativity in a Modern Physics course is enough to deal with this misconception.

Additionally, the quasi-rigorous nature of the middle third of a Modern Physics course does not expose the students to any useful mathematical or physical tools. This third involves a qualitative, hand-wavy discussion of the history of the atom that leads to the students quoting results on exams. At the very least, the Modern Physics course could be restructured in such a way that this qualitative discussion of the history of the atom can be abbreviated, allowing more time for exposure to the actual modern topics – relativistic mechanics and quantum mechanics – discussed in the remainder of the course. This would better prepare the students for their future studies [10,11].

2.4 Conclusion and Future Prospectus

This work proposes a simple, digestible approach to exposing undergraduate physics students to the concept of acceleration in special relativity. In doing so, this conversation lends itself to a more rigorous and honest conversation of both special and general relativity. Furthermore, the use of a case-study method – with proven utility for preventing the development of misconceptions – is proposed with an example of how this could be performed. Moving forward, it is certainly prudent to test the efficacy of this method on a large scale in preventing misconceptions regarding acceleration in special relativity.

2.5 References

[1] J. W. Smyth, *Aust. Jour. of Edu.* 24, 225 (1980)

[2] A. K. Griffiths and K. R. Preston, *Jour. Of Resear. in Sci. Teach.* 29, 611 (1992)

[3] T. Plotz, *Phys. Edu.* 52, 014004 (2016)

[4] H. N. Bani-Salameh, *Phys. Edu.* 52, 014003 (2016)

[5] N. Fotou and I. Abrahams, *Phys. Edu.* 51, 044003 (2016)

[6] *N.A., Scientific Teaching Reconsidered: A Handbook,* Chap. 4 (National Research Council, 1997)

[7] D. Brown and J. Clement, *Institut für die Pädagogik der Naturwissenschaften an der Universität Kiel,* pp. 380-397 (1991)

[8] G. A. Lieberman, L. L. Hoody, *State Education and Environment Roundtable* (1998)

[9] Y. Ayyildiz and L. Tarhan, *Chem. Edu. Resear. and Pract.*, 14, 408 (2013)

[10] P. M. Sadler and R. H. Tai, *Success in Introductory College Physics: The role of High School Preparation.* 111-136, (John Wiley & Sons, Inc., 2001)

[11] L. F. Santos, *Jour. of Edu. and pract.* 8, 159 (2017)

[12] S. Carroll, Spacetime *and Geometry: An Introduction to General Relativity*, 1st ed., Chap. 1, Sec. 2 (Pearson, 2003).

2.6   Appendix B

Relativistic Mechanics Worksheet
World-Lines and Accelerated Frames

At this point it is expected that you have had a discussion regarding Minkowski space, world-lines, spacetime diagrams, and of the twin paradox. The problem at the heart of the twin paradox lies at the points of acceleration. It should be intuitively obvious that going from a speed of 0 to that of 0.8c in an instant would turn a human into a pancake at the back of their space ship. In the classical presentation of the twin paradox, this "pancaking" happens three times – when the traveling twin leaves earth, when the traveling twin goes from $+0.8c$ to $-0.8c$, and when it accelerates from -0.8c to 0c back on earth.

Moving forward, it would be prudent to show that accelerating at a reasonable acceleration, say 9.8 m/s, would still allow for significant time dilation to be experienced by the twins. For convenience sake we will, of course, ignore the energy required to maintain that acceleration. So, let's start with what we know about worldliness and the spacetime metric. We know the spacetime metric to be:

$$ds' = ds$$

$$\implies c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

(1)

Where τ is the proper time, t is the time in the moving frame, and x, y, and z are the spatial coordinates. In the case of the twin paradox, the traveling twin only moves in the x-direction, simplifying the metric to:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 \tag{2}$$

Let's take a closer look at this metric in the classical presentation of the twin paradox.
1. Show that solving this differential equation for τ in terms of t gives Equation 3.

$$\Delta\tau = \int_{t_i}^{t_f} \sqrt{1 - \frac{v^2}{c^2}}\, dt \tag{3}$$

Having arrived at our general expression for τ as a function of time, let's apply this to the classical twin paradox problem. Considering the spacetime diagram, the twin instantaneously accelerates to some fraction of the speed of light, v_1, in the x-direction at time t_0. At some time t_1, the twin instantaneously reverses their velocity. Finally, at time t_2, the twin instantaneously comes to rest back on earth.

2. Construct the spacetime diagram for this problem and check your answer against your peers.

3. Now apply Equation 3 to this problem to find the expression for the change in proper time. (Hint: start by treating the different time intervals individually).

You should have arrived at

$$\Delta\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v_1^2}{c^2}}\, dt + \int_{t_1}^{t_2} \sqrt{1 - \frac{v_2^2}{c^2}}\, dt \tag{4}$$

In this example, the twin travels at a constant velocity during each time interval. As a result, Equation 4 becomes:

$$\Delta\tau = \sqrt{1 - \frac{v_1^2}{c^2}} \int_{t_0}^{t_1} dt + \sqrt{1 - \frac{v_2^2}{c^2}} \int_{t_1}^{t_2} dt \tag{5}$$

Furthermore, since the square of each velocity is equivalent this can be further simplified. We will simultaneously recognize that the constant in each integral in Equation 5 is the inverse of the Lorentz factor.

$$\Delta\tau = \sqrt{1 - \frac{v_1^2}{c^2}} \int_{t_0}^{t_2} dt$$

$$\implies \Delta\tau = \gamma^{-1}\Delta t \tag{6}$$

$$\implies \gamma = \frac{\Delta t}{\Delta\tau}$$

This brings us to the formal relationship between proper time, time, and the Lorentz factor. This is part of the utility of this classical presentation of the twin paradox. It allows the problem to be presented straightforwardly as an algebra problem. As we will show next, any problem involving non-inertial twins leads to more complicated integration. As a result, many Modern Physics courses do not discuss this topic. Unfortunately, this often leads to a misconception that special relativity is not capable of dealing with non-inertial problems. We will deal with this misconception head-on.

Non-inertial twin paradox in K:

We will now consider the two twins in a slightly different scenario. We will define the observer on the earth as being in the reference frame K, and the spaceship twin to be in the reference frame K'. In this situation, the spaceship is passing earth at some time, t_0, with a velocity $v_0 = 0.8c$. At t_0, as the twin passes the earth and synchronizes he watch with his twin, he begins accelerating in the negative x-direction at $g = 9.8$ m/s^2 (a non-pancaking velocity). At some time, t_1, the twin passes the earth once again at $v_1 = -v_0$ and continues in the negative x-direction. The spacetime diagram for this problem is given below in Figure 2 (see Figure 2.2).

4. Using Equation 3, show that kinematics implies that the expression for τ as a function of time implies Equation 7.

$$\Delta\tau = \int_{t_i}^{t_f} \sqrt{1 - \frac{(v_i - gt)^2}{c^2}}\, dt \tag{7}$$

This problem is cleaner in some sense, because it only requires one-time interval; however, it is obvious that this integral has now become more difficult because the velocity is now a function of time and the integration is more involved. A good approach is to use trigonometric substitution, but explicitly solving this problem is left to the discretion of the worksheet facilitator.

$$\tau(t) = \left[\frac{\arcsin\left(\frac{v_i - gt}{c}\right) + \left(\frac{v_i - gt}{c}\right)\left(1 - \left(\frac{v_i - gt}{c}\right)^2\right)^{1/2}}{2} \right] \tag{8}$$

This problem can also be solved in the frame of S', but one must be careful in setting up the acceleration in S'.

This example shows an interesting property of the spacetime metric. In effect, we were integrating over the world line of the spaceship twin as seen from the reference frame of the earth twin. However, the value you will obtain for the proper time of the moving twin is smaller than that of stationary twin. This is very counterintuitive, and why we must be careful by discussing distances in terms of the spacetime interval when relativity is relevant.

As a take home problem, consider a different example: let the spaceship twin start on earth, accelerating first away from the earth at g until some time, t_1, at which point the direction of the acceleration is reversed. The twin continues undergoing this acceleration until some time t_2, where the acceleration reverts back to the original acceleration direction, landing the twin stationary back on earth at time t_3. Construct the spacetime diagram for this example, and set up your expression for $\Delta\tau$ in this situation (hint: be careful how you construct your integral bounds).